

# An extremal problem on potentially $K_{p_1, p_2, \dots, p_t}$ -graphic sequences \*

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## Abstract

A sequence  $S$  is potentially  $K_{p_1, p_2, \dots, p_t}$  graphical if it has a realization containing a  $K_{p_1, p_2, \dots, p_t}$  as a subgraph, where  $K_{p_1, p_2, \dots, p_t}$  is a complete  $t$ -partite graph with partition sizes  $p_1, p_2, \dots, p_t$  ( $p_1 \geq p_2 \geq \dots \geq p_t \geq 1$ ). Let  $\sigma(K_{p_1, p_2, \dots, p_t}, n)$  denote the smallest degree sum such that every  $n$ -term graphical sequence  $S$  with  $\sigma(S) \geq \sigma(K_{p_1, p_2, \dots, p_t}, n)$  is potentially  $K_{p_1, p_2, \dots, p_t}$  graphical. In this paper, we prove that  $\sigma(K_{p_1, p_2, \dots, p_t}, n) \geq 2[(2p_1 + 2p_2 + \dots + 2p_t - p_1 - p_2 - \dots - p_i - 2)n - (p_1 + p_2 + \dots + p_t - p_i)(p_i + p_{i+1} + \dots + p_t - 1) + 2]/2$  for  $n \geq p_1 + p_2 + \dots + p_t, i = 2, 3, \dots, t$ .

**Key words:** graph; degree sequence; potentially  $K_{p_1, p_2, \dots, p_t}$ -graphic sequence

**AMS Subject Classifications:** 05C07, 05C35

## 1 Introduction

If  $S = (d_1, d_2, \dots, d_n)$  is a sequence of non-negative integers, then it is called graphical if there is a simple graph  $G$  of order  $n$ , whose degree sequence  $(d(v_1), d(v_2), \dots, d(v_n))$  is precisely  $S$ . If  $G$  is such a graph then  $G$  is said to realize  $S$  or be a realization of  $S$ . A graphical sequence  $S$  is potentially  $H$  graphical if there is a realization of  $S$  containing  $H$  as a subgraph, while  $S$  is forcibly  $H$  graphical if every realization of  $S$  contains  $H$  as a subgraph. Let  $\sigma(S) = d(v_1) + d(v_2) + \dots + d(v_n)$ , and  $[x]$  denote the largest integer less than or equal to  $x$ . If  $G$  and  $G_1$  are graphs, then

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$G \cup G_1$  is the disjoint union of  $G$  and  $G_1$ . If  $G = G_1$ , we abbreviate  $G \cup G_1$  as  $2G$ . Let  $K_k$ , and  $C_k$  denote a complete graph on  $k$  vertices, and a cycle on  $k$  vertices, respectively. Let  $K_{p_1, p_2, \dots, p_t}$  denote a complete  $t$ -partite graph with partition sizes  $p_1, p_2, \dots, p_t$  ( $p_1 \geq p_2 \geq \dots \geq p_t \geq 1$ ).

Given a graph  $H$ , what is the maximum number of edges of a graph with  $n$  vertices not containing  $H$  as a subgraph? This number is denoted  $ex(n, H)$ , and is known as the Turán number. This problem was proposed for  $H = C_4$  by Erdős [3] in 1938 and in general by Turán [11]. In terms of graphic sequences, the number  $2ex(n, H) + 2$  is the minimum even integer  $l$  such that every  $n$ -term graphical sequence  $S$  with  $\sigma(S) \geq l$  is forcibly  $H$  graphical. Here we consider the following variant: determine the minimum even integer  $l$  such that every  $n$ -term graphical sequence  $S$  with  $\sigma(S) \geq l$  is potentially  $H$  graphical. We denote this minimum  $l$  by  $\sigma(H, n)$ . Erdős, Jacobson and Lehel [4] showed that  $\sigma(K_k, n) \geq (k-2)(2n-k+1)+2$  and conjectured that equality holds. They proved that if  $S$  does not contain zero terms, this conjecture is true for  $k=3$ ,  $n \geq 6$ . The conjecture is confirmed in [5], [7], [8], [9] and [10].

Gould, Jacobson and Lehel [5] also proved that  $\sigma(pK_2, n) = (p-1)(2n-2)+2$  for  $p \geq 2$ ;  $\sigma(C_4, n) = 2\lceil \frac{3n-1}{2} \rceil$  for  $n \geq 4$ . Yin and Li [12] gave sufficient conditions for a graphic sequence being potentially  $K_{r,s}$ -graphic, and determined  $\sigma(K_{r,r}, n)$  for  $r=3, 4$ . Lai [6] proved that  $\sigma(K_4 - e, n) = 2\lceil \frac{3n-1}{2} \rceil$  for  $n \geq 7$ . In this paper, we prove that  $\sigma(K_{p_1, p_2, \dots, p_t}, n) \geq 2[\frac{((2p_1+2p_2+\dots+2p_t-p_1-p_2-\dots-p_i-2)n-(p_1+p_2+\dots+p_t-p_i)(p_i+p_{i+1}+\dots+p_t-1)+2)}{2}]$  for  $n \geq p_1+p_2+\dots+p_t$ ,  $i=2, 3, \dots, t$ .

## 2 Main results.

**Theorem 1.**  $\sigma(K_{p_1, p_2, \dots, p_t}, n) \geq 2[\frac{((2p_1+2p_2+\dots+2p_t-p_1-p_2-\dots-p_i-2)n-(p_1+p_2+\dots+p_t-p_i)(p_i+p_{i+1}+\dots+p_t-1)+2)}{2}]$  for  $n \geq p_1+p_2+\dots+p_t$ ,  $i=2, 3, \dots, t$ .

**Proof.** We first consider  $p_1+p_2+\dots+p_i-2p_i$  is even. If  $n-p_i-p_{i+1}-\dots-p_t+1$  is even, let  $n-p_i-p_{i+1}-\dots-p_t+1=2m$ , By Theorem 11.5.9 of [2],  $K_{2m}$  is the union of one 1-factor  $M$  and  $m-1$  spanning cycles  $C_1^1, C_2^1, \dots, C_{m-1}^1$ . Let

$$H = C_1^1 \cup C_2^1 \cup \dots \cup C_{\frac{p_1+p_2+\dots+p_i-2p_i}{2}}^1 + K_{p_i+p_{i+1}+\dots+p_t-1}$$

Then  $H$  is a realization of  $((n-1)^{p_i+p_{i+1}+\dots+p_t-1}, (p_1+p_2+\dots+p_t-p_i-1)^{n-p_i-p_{i+1}-\dots-p_t+1})$ . Since  $((p_1+p_2+\dots+p_t-p_t)^{p_t}, (p_1+p_2+\dots+p_t-p_{t-1})^{p_{t-1}}, \dots, (p_1+p_2+\dots+p_t-p_1)^{p_1})$  is the degree sequence of  $K_{p_1, p_2, \dots, p_t}$ ,  $((n-1)^{p_i+p_{i+1}+\dots+p_t-1}, (p_1+p_2+\dots+p_t-p_i-1)^{n-p_i-p_{i+1}-\dots-p_t+1})$  is not potentially  $K_{p_1, p_2, \dots, p_t}$  graphic. Thus  $\sigma(K_{p_1, p_2, \dots, p_t}, n) \geq (p_i+p_{i+1}+\dots+p_t-1)(n-1)+(p_1+p_2+\dots+p_t-p_i-1)(n-p_i-p_{i+1}-\dots-p_t+1)+2 = 2[\frac{((2p_1+2p_2+\dots+2p_t-p_1-p_2-\dots-p_i-2)n-(p_1+p_2+\dots+p_t-p_i)(p_i+p_{i+1}+\dots+p_t-1)+2)}{2}]$ . Next, If  $n-p_i-p_{i+1}-\dots-p_t+1$  is odd, let  $n-p_i-p_{i+1}-\dots-p_t+1=2m+1$ , By Theorem 11.5.9 of [2],  $K_{2m+1}$  is the union of  $m$  spanning cycles  $C_1^1, C_2^1, \dots, C_m^1$ . Let

$$H = C_1^1 \cup C_2^1 \cup \dots \cup C_{\frac{p_1+p_2+\dots+p_i-2p_i}{2}}^1 + K_{p_i+p_{i+1}+\dots+p_t-1}$$

Then  $H$  is a realization of  $((n-1)^{p_i+p_{i+1}+\dots+p_t-1}, (p_1+p_2+\dots+p_t-p_i-1)^{n-p_i-p_{i+1}-\dots-p_t+1})$ , and we are done as before. This completes the discussion for  $p_1 + p_2 + \dots + p_i - 2p_i$  is even.

Now we consider  $p_1 + p_2 + \dots + p_i - 2p_i$  is odd. If  $n - p_i - p_{i+1} - \dots - p_t + 1$  is even, let  $n - p_i - p_{i+1} - \dots - p_t + 1 = 2m$ , By Theorem 11.5.9 of [2],  $K_{2m}$  is the union of one 1-factor  $M$  and  $m - 1$  spanning cycles  $C_1^1, C_2^1, \dots, C_{m-1}^1$ . Let

$$H = M \cup C_1^1 \cup C_2^1 \cup \dots \cup C_{\frac{p_1+p_2+\dots+p_i-2p_i-1}{2}}^1 + K_{p_i+p_{i+1}+\dots+p_t-1}$$

Then  $H$  is a realization of  $((n-1)^{p_i+p_{i+1}+\dots+p_t-1}, (p_1+p_2+\dots+p_t-p_i-1)^{n-p_i-p_{i+1}-\dots-p_t+1})$ , and we are done as before. Next, If  $n - p_i - p_{i+1} - \dots - p_t + 1$  is odd, let  $n - p_i - p_{i+1} - \dots - p_t + 1 = 2m + 1$ , By Theorem 11.5.9 of [2],  $K_{2m+1}$  is the union of  $m$  spanning cycles  $C_1^1, C_2^1, \dots, C_m^1$ . Let

$$C_1^1 = x_1 x_2 \dots x_{2m+1} x_1$$

$$H = (C_1^1 \cup C_2^1 \cup \dots \cup C_{\frac{p_1+p_2+\dots+p_i-2p_i+1}{2}}^1 + K_{p_i+p_{i+1}+\dots+p_t-1})$$

$$- \{x_1 x_2, x_3 x_4, \dots, x_{2m-1} x_{2m}, x_{2m+1} x_1\}$$

Then  $H$  is a realization of  $((n-1)^{p_i+p_{i+1}+\dots+p_t-1}, (p_1+p_2+\dots+p_t-p_i-1)^{n-p_i-p_{i+1}-\dots-p_t}, (p_1 + p_2 + \dots + p_t - p_i - 2)^1)$ . It is easy to see that  $((n-1)^{p_i+p_{i+1}+\dots+p_t-1}, (p_1 + p_2 + \dots + p_t - p_i - 1)^{n-p_i-p_{i+1}-\dots-p_t}, (p_1 + p_2 + \dots + p_t - p_i - 2)^1)$  is not potentially  $K_{p_1, p_2, \dots, p_t}$  graphic. Thus  $\sigma(K_{p_1, p_2, \dots, p_t}, n) \geq (p_i + p_{i+1} + \dots + p_t - 1)(n - 1) + (p_1 + p_2 + \dots + p_t - p_i - 1)(n - p_i - p_{i+1} - \dots - p_t) + (p_1 + p_2 + \dots + p_t - p_i - 2) + 2 = 2[(2p_1 + 2p_2 + \dots + 2p_t - p_1 - p_2 - \dots - p_i - 2)n - (p_1 + p_2 + \dots + p_t - p_i)(p_i + p_{i+1} + \dots + p_t - 1) + 2]/2$ . This completes the discussion for  $p_1 + p_2 + \dots + p_i - 2p_i$  is odd, and so finishes the proof of Theorem 1.

By [4],[5],[7],[8],[9] and [10]. The equality holds for  $p_1 = p_2 = \dots = p_t = 1, i = 2$ .

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